Constant mean curvature surfaces via integrable dynamical system

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Abstract: It is shown that the equation which describes constant mean curvature surfaces via the generalized Weierstrass–Enneper inducing has Hamiltonian form. Its simplest finite–dimensional reduction is the integrable Hamiltonian system with two degrees of freedom. This finite-dimensional system admits S^1-action and classes of S^1-equivalence of its trajectories are in one-to-one correspondence with different helicoidal constant mean curvature surfaces. Thus the interpretaion of well-known Delaunay and do Carmo–Dajzcer surfaces via integrable finite–dimensional Hamiltonian system is established.

Surfaces, interfaces, fronts, and their dynamics are key ingredients in a number of interesting phenomena in physics. They are surface waves, growth of crystal, deformation of membranes, propagation of flame fronts, many problems of hydrodynamics connected with motion of boundaries between regions of differ densities and viscosities (see, e.g., [1, 2]). Quantum field theory and statistical physics are the important customers of surfaces too (see [3, 4]).

Mean curvature plays special role among the characteristics of surfaces and their dynamics in several problems both in physics and mathematics (see, e.g., [5, 6]). Surfaces of constant mean curvature have been studied intensively during last years (see, e.g., [7, 8, 9]).

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In present paper we discuss a new approach for construction of constant mean curvature surfaces. This method is based on the generalized Weierstrass–Enneper inducing ([10, 11, 12]). It allows to generate constant mean curvature surfaces via integrable dynamical system with two degrees of freedom. The relation between the trajectories of different types and surfaces of different types is established.

The generalization of the Weierstrass–Enneper formulas for inducing minimal surfaces have been proposed in [10] (see also [11]) and rediscovered in different but equivalent form in connection with integrable nonlinear equations in [12]. We will use notation and formulae from [12].

We start with the linear system

$$\begin{aligned} \psi_{1z} &= p\psi_2, \\ \psi_{2\bar{z}} &= -p\psi_1, \end{aligned} \tag{1}$$

where $p(z, \bar{z})$ is a real function, ψ_1 and ψ_2 are, in general, complex functions of the complex variable z, and bar denotes the complex conjugation. By using of the solution of (1), one introduces the variables $(X^1(z, \bar{z}), X^2(z, \bar{z}), X^3(z, \bar{z}))$ as follows

$$\begin{aligned} X^1 + iX^2 &= 2i \int_{z_0}^{z} (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \\ X^1 - iX^2 &= 2i \int_{z_0}^{z} (\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \\ X^3 &= -2 \int_{z_0}^{z} (\psi_2 \bar{\psi}_1 dz' + \psi_1 \bar{\psi}_2 d\bar{z}'). \end{aligned} \tag{2}$$

In virtue of (1) integrals (2) do not depend on the choice of the curve of integration.

Then one treats $z, \bar{z}$ as local coordinates on a surface and (X^1, X^2, X^3) as coordinates of its immersion in $\mathbf{R}^3$. Formulae (2) induce a surface in $\mathbf{R}^3$ via the solutions of system (1). By using of the well-known formulae, one finds the first fundamental form

$$\tilde{\Omega} = 4(|\psi_1|^2 + |\psi_2|^2)^2 dz d\bar{z} \tag{3}$$

and Gaussian (K) and mean (H) curvatures

$$K = -\frac{(\log(|\psi_1|^2 + |\psi_2|^2))_{z\bar{z}}}{(|\psi_1|^2 + |\psi_2|^2)^2}, \quad H = \frac{p(z, \bar{z})}{|\psi_1|^2 + |\psi_2|^2}. \tag{4}$$

This type of inducing of surfaces is the generalization of the well-known Weierstrass–Enneper inducing of minimal surfaces. Indeed, minimal surfaces ($H \equiv 0$) correspond to $p \equiv 0$ and in this case formulae (2) in terms of functions $\psi = \frac{1}{\sqrt{2}}\psi_2$ and $\phi = \frac{1}{\sqrt{2}}\bar{\psi}_1$ are reduced to those of Weierstrass–Enneper.

In this paper we will consider the case of constant mean curvature surfaces. In this case $p = H(|\psi_1|^2 + |\psi_2|^2)$ where $H = \mathit{const}$ and system (1) is reduced to the following

$$\psi_{1t} - i\psi_{1x} = 2H(|\psi_1|^2 + |\psi_2|^2)\psi_2,$$

$$\psi_{2t} + i\psi_{2x} = -2H(|\psi_1|^2 + |\psi_2|^2)\psi_1, \tag{5}$$

where $z = t + ix$.

First we note that system (5) has four obvious real integrals of motion (independent on t):

$$C_+ = \int dx(\psi_1^2 + \psi_2^2 + \bar{\psi}_1^2 + \bar{\psi}_2^2),$$

$$C_- = \frac{1}{i}\int dx(\psi_1^2 + \psi_2^2 - \bar{\psi}_1^2 - \bar{\psi}_2^2),$$

$$P = \int dx(\psi_{1x}\bar{\psi}_2 - \bar{\psi}_1\psi_{2x}), \tag{6}$$

$$\mathcal{H} = \int dx\{\frac{i}{2}(\psi_{1x}\bar{\psi}_2 + \bar{\psi}_1\psi_{2x}) + H(|\psi_1|^2 + |\psi_2|^2)^2\}.$$

Then this system is Hamiltonian, i.e. it can be represented in the form

$$\psi_{1t} = \{\psi_1, \mathcal{H}\}, \psi_{2t} = \{\psi_2, \mathcal{H}\} \tag{7}$$

where the Hamiltonian $\mathcal{H}$ is given by (6) and the Poisson bracket $\{,\}$ is of the form

$$\{F_1, F_2\} = \int dx\{(\frac{\delta F_1}{\delta\psi_1}\frac{\delta F_2}{\delta\bar{\psi}_2} - \frac{\delta F_1}{\delta\psi_2}\frac{\delta F_2}{\delta\bar{\psi}_1}) - (\frac{\delta F_2}{\delta\psi_1}\frac{\delta F_1}{\delta\bar{\psi}_2} - \frac{\delta F_2}{\delta\psi_2}\frac{\delta F_1}{\delta\bar{\psi}_1})\}. \qquad (8).$$

The corresponding symplectic form is

$$\Omega = d\psi_1 \wedge d\bar{\psi}_2 + d\bar{\psi}_1 \wedge d\psi_2$$

and the Lagrangian is given by the following formula

$$\mathcal{L} = \psi_1 \bar{\psi}_{2z} - \bar{\psi}_{1\bar{z}} \psi_2 + \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2)^2.$$

Thus formula (2) establishes the correspondence between the trajectories of the infinite–dimensional Hamiltonian system (5) and surfaces of constant mean curvatures.

Let us put

$$H \neq 0$$

to omit the discussion of minimal surfaces.

Let us also restrict ourselves to the particular case of this inducing with $p = p(t)$. It is not difficult to show that under this constraint the only admissible solutions, of system (5), which are representable by finite sums of terms of the type $f(t) \exp i\rho x$ are of the form

$$\psi_1 = r(t) \exp(i\lambda x), \psi_2 = s(t) \exp(i\lambda x), \tag{9}$$

where $\lambda(\neq 0)$ is real parameter and $r(t) = p_1 + ip_2$ and $s(t) = q_1 + iq_2$ are complex-valued functions. System (5) in these variables has the following form

$$\begin{aligned} r_t + \lambda r - 2H(|r|^2 + |s|^2)s = 0, \\ s_t - \lambda s + 2H(|r|^2 + |s|^2)r = 0, \end{aligned} \tag{10}$$

or equivalent system of four equations in terms of real and imaginary parts of r and s. It has the Hamiltonian form

$$\frac{\partial p_i}{\partial t} = \{p_i, \mathcal{H}_0\}_0, \frac{\partial q_j}{\partial t} = \{q_j, \mathcal{H}_0\}_0, \quad i, j = 1, 2,$$

with the Hamiltonian function

$$\mathcal{H}_0 = \frac{H}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - \lambda(p_1 q_1 + p_2 q_2)$$

and with respect to the usual Poisson brackets $\{,\}_0$ generated by the simplectic form

$$\Omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

It is easy to notice that the Hamiltonian function $\mathcal{H}_0$ can be obtained from $\mathcal{H}$ by using of the finite dimensional reduction (9). Hamiltonian system (10) has another first integral

$$M = p_1 q_2 - p_2 q_1$$

which is in involution with the Hamiltonian $\mathcal{H}_0$ and moreover these first integrals are functionally independent everywhere except the zero ($p_i = q_j = 0$). Thus we conclude that system (10) is integrable.

This system is not only integrable but also S^1-symmetric. Its Hamiltonian, the additional first integral M and the Poisson structure are preserved by the following S^1-action:

$$\begin{cases} p_1 \to p_1 \cos\phi - p_2 \sin\phi \\ p_2 \to p_1 \sin\phi + p_2 \cos\phi \end{cases}, \quad \begin{cases} q_1 \to q_1 \cos\phi - q_2 \sin\phi \\ q_2 \to q_1 \sin\phi + q_2 \cos\phi \end{cases}. \tag{11}$$

Let us assume without loss of generality that

$$\lambda = H = \frac{1}{2}.$$

Formulae (2) obtain the following form

$$\begin{aligned} X^1 = -2\int\{[(p_1^2 + q_1^2 - p_2^2 - q_2^2)\cos x - 2(p_1p_2 + q_1q_2)\sin x]dx \\ +[2(q_1q_2 - p_1p_2)\cos x + (q_1^2 + p_2^2 - q_2^2 - p_1^2)\sin x]dt\}, \\ X^2 = 2\int\{[2(p_1p_2 + q_1q_2)\cos x + (p_1^2 + q_1^2 - p_2^2 - q_2^2)\sin x]dx \\ +[(p_1^2 + q_2^2 - p_2^2 - q_1^2)\cos x + 2(q_1q_2 - p_1p_2)\sin x]dt\}, \\ X^3 = -4\int\{(p_1q_1 + p_2q_2)dt - (p_1q_2 - p_2q_1)\}dx. \end{aligned} \tag{12}$$

Trajectories of Hamiltonian system (10) which are different modulo symmetry (11) describe different constant mean curvature surfaces by using of formulas (12). It also follows from (12) that these surfaces are invariant under the following helicoidal transform:

$$\begin{cases} X^1 \to X^1 \cos\tau - X^2 \sin\tau \\ X^2 \to X^1 \sin\tau + X^2 \cos\tau \\ X^3 \to X^3 + 4M\tau \end{cases}, \tag{13}$$

and the restriction, of this transform, to the surface coincides with the shift of $Imz = x: \quad x \to x + \tau$.

We see that if $M = 0$ then we obtain a surface of revolution. All these surfaces are equivalent modulo (11) to surfaces with $p_2 \equiv q_2 \equiv 0$. It is not complicated to give a qualitative analysis of the behaviour of the restriction of (10) onto this plane. This vector field has three zeros at points $(0,0)$ and $\pm\frac{1}{2}, \pm\frac{1}{2})$. The second ones correspond to cylinders of revolution. At these points the Hamiltonian $\mathcal{H}_0$ is equal to $-\frac{1}{32}$. These points are bounded by cycles on which Hamiltonian is negative but more than $-\frac{1}{32}$ and which correspond to unduloids (i.e., the Delaunay surfaces which are embedded into $\mathbf{R}^3$ and differ from cylinder and round sphere). Hamiltonian vanishes at the zero point and two separatrices which come from $(0,0)$ and arrives to it. These separatrices correspond to round sphere with a pair of truncated points and bound a domain where Hamiltonian is negative. The domain $\mathcal{H}_0 > 0$ is fibered by cycles of Hamiltonian system (10) and these cycles correspond to nodoids (i.e., Delaunay surfaces which have selfintersections).

Thus we obtain very natural Hamiltonian interpretation for the well-known family of Delaunay surfaces ([13]).

In the same manner it is shown that the full family of surfaces which corresponds to solutions of (10) with $M \neq 0$ coincides with the family of helicoidal surfaces of constant mean curvature which were constructed in [14].

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